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STOCHASTIC RELATIONS IN LINEAR FILTERING AND IDENTIFICATION PROBLEMS

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OUTLINE

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Abstract

In this paper are presented some computing methods of the variance σ^2_{y} and the covariance σ_{xy} , the auto-correlation function $A_y(\theta)$, the cross-correlation $A_{xy}(\theta)$ of the output fluctuation y in control systems. The systems are assumed linear and to have a single pair of input and output. The auto-correlation function $A_x(\theta)$ of the input x is to be expressed by an exponential function. The transfer function $g(p)$ of the system is to be expressed by a fraction $N(p)/D(p)$, the denominator of which being a polynomial of a high order, higher of one than that of the numerator.

Applications of these computing methods are given for some special cases.

It is also shown how to extend the use of these computing methods to the parameter identification of the system transfer function.

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1. INTRODUCTION

The purpose of this report is to present some general relations for the computation of the auto-correlation and the cross-correlation function of a linear system, with a single pair of input and output, when the transfer function of the system is given by a fraction, the denominator of which being a polynomial of a high order, higher of one than that of the numerator.

This report shows how to use these relations for the identification of the parameters of the transfer function, with the use of auto-correlation, cross-correlation, variance and covariance, computed from measured samples of the input and the output of the system.

Let us consider a linear system with input x, output y and the transfer function $g(p) = N(p)/D(p)$ to be identified.

If the auto-correlation $A_{xx}(\theta)$ of the input is known, the auto-correlation $A_{yy}(\theta)$, the cross-correlation $A_{xy}(\theta)$, meansquare value of the output, or the variance σ_y^2 and the covariance σ_{xy} can be calculated with the following integrals (1 to 7)

$$A_{yy}(\theta) = \int_0^\infty A_{xx}(u+\theta)A_g(u)\,du + \int_0^\infty A_{xx}(u-\theta)A_g(u)\,du$$

$$\text{with } A_g(\theta) = \int_0^\infty g(t)\,g(t+\theta)\,dt$$

$$A_{xy}(\theta) = \int_0^\infty A_{xx}(\theta-\lambda)g(\lambda)\,d\lambda$$

$$\sigma_y^2 = A_{yy}(0) = \int_0^\infty A_{xx}(\theta)A_g(\theta)\,d\theta$$

$$\sigma_{xy} = A_{xy}(0) = \int_0^\infty A_{xx}(\theta)g(\theta)\,d\theta$$

If the auto-correlation $A_{xx}(\theta)$ of the input can be described by an exponential function or by a sum of exponentials, it is possible to obtain general expressions for the auto-correlation $A_{yy}(\theta)$, the cross-correlation $A_{xy}(\theta)$, the variance σ_y^2 and the covariance σ_{xy} of the output.

2. GENERAL RELATION FOR THE STOCHASTIC FILTERING

We assume that $g(p) = \dfrac{N(p)}{D(p)}$

with $g(0) = 1$

$N(p)$ = polynomial of an order less than $n - 1$

$D(p)$ = polynomial for an order n

This transfer function can also be written :

$$g(p) = \sum_{i=1}^{n} \frac{\alpha_i}{p-p_i}$$

$$\text{with } \alpha_i = \frac{N(p_i)}{\left|\frac{d}{dp}D(p)\right|_{p=p_i}} = (-1)^n \prod (p_j) \frac{N(p_i)}{\prod_{j=1}^{n}(p-p_j)}$$

(with $j \neq i$)

α_i = residues of $g(p)$

p_i = roots of the equation $D(p) = 0$

We obtain, with $A_{xx}(\theta) = \sigma_x^2\, e^{k|\theta|}$:

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = g(k)g(-k)e^{k|\theta|} + 2k\sum_{i=1}^{n} \frac{\alpha_i\, g(-p_i)\, e^{p_i|\theta|}}{p_i^2 - k^2}$$

$$\text{If } g(p) = \frac{N(p)}{b_n \prod^n (p-p_i)}$$

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = \frac{1}{b_n^2}\left[\frac{N(k)\,N(-k)}{\prod_{i=1}^{n}(p_i^2-k^2)}\, e^{k|\theta|} + (-1)^n k \sum_{i=1}^{n} \frac{N(p_i)\,N(-p_i)\, e^{p_i|\theta|}}{p_i(p_i^2-k^2)\prod_{j=1}^{n}(p_i^2-p_j^2)}\right]$$

$$\frac{\sigma_y^2}{\sigma_x^2} = -2\sum_{j=1}^{n} \frac{\alpha_j\, g(-p_j)}{p_j + k}$$

with $j \neq i$ and $j \neq k$

Special case of complex roots

The given formula is true for real or complex roots. In the last case, if we put $p_k = x_k + i\, y_k$, the auto-correlation $A_{yy}(\theta)$ becomes

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = g(k)\,g(-k)\,e^{k|\theta|} + 4k \sum_{j=1}^{m} e^{x_k|\theta|}\left[X_k \cos y_k|\theta| + Y_k \sin y_k|\theta|\right] + 2k \sum \frac{\alpha_i\, g(-p_i)}{p_i^2 - k^2}\, e^{p_i|\theta|}$$

with X_k = real part of $\dfrac{\alpha_k\, g(-p_k)}{p_k^2 - k^2}$

Y_k = imaginary part of $\dfrac{\alpha_k\, g(-p_k)}{p_k^2 - k^2}$

Special case of $p_1 = k$

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = \alpha_1 g(-p_1)\cdot|\theta|\, e^{p_1|\theta|} + B\, e^{p_1|\theta|} + 2k\sum_{i=2}^{n} \frac{\alpha_i\, g(-p_i)}{p_i^2 - p_1^2}\, e^{p_i|\theta|}$$

with
$$B = \alpha_1 g(-p_1)\left\{-\frac{1}{2p_1} + \frac{\left|\frac{d}{ds}N(s)\right|_{s=p_1}}{N(p_1)} + \frac{\left|\frac{d}{ds}N(-s)\right|_{s=p_1}}{N(-p_1)} - \frac{\left|\frac{d}{ds}\prod_{j=2}^{n}(s-p_j)\right|_{s=p_1}}{\prod_{j=2}^{n}(p_1-p_j)} - \frac{\left|\frac{d}{ds}D(-s)\right|_{s=p_1}}{D(-p_1)}\right\}$$

Special case of multiple roots of r order

We suppose that $g(p)$ has a root p_1 of r order

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = g(k)\,g(-k)\,e^{k|\theta|} + 2k\sum_{i=r+1}^{n} \frac{\alpha_i\, g(-p_i)\, e^{p_i|\theta|}}{p_i^2 - k^2} + \sum_{j=1}^{r} B_j\, \frac{|\theta|^{j-1}}{(j-1)!}\, e^{p_1|\theta|}$$

with
$$B_j = \frac{1}{(r-j)!}\left\{\frac{d}{ds^{r-j}}\left[\frac{2k}{s^2-k^2}\, g(-s)\, \frac{N(s)}{\prod_{i=r+1}^{n}(s-p_i)}\right]\right\}_{s=p_1}$$

$$\frac{\sigma_y^2}{\sigma_x^2} = g(k)\,g(-k) + 2k\sum_{i=r+1}^{n} \frac{\alpha_i\, g(-p_i)}{p_1 - k^2} + \frac{1}{(r-1)!}\,\frac{d}{ds^{r-1}}\left[\frac{2k}{s^2-k^2}\, g(-s)\, \frac{N(s)}{\prod_{i=r+1}^{n}(s-p_i)}\right]_{s=p_1}$$

Special case where the auto-correlation function of the input is given by a sum of exponential functions

We assume that $A_{xx}(\theta) = \sigma_x^2 \sum_{m=1}^{t} a_m\, e^{k_m|\theta|}$

with $\sum_{m=1}^{t} a_m = 1$

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = \sum_{m=1}^{t} g(k_m)\,g(-k_m)\,a_m\, e^{k_m|\theta|} + 2k\sum_{i=1}^{n}\alpha_i\, g(-p_i)\, e^{p_i|\theta|} \sum_{m=1}^{t} \frac{a_m}{p_i^2 - k_m^2}$$

$$\frac{\sigma_y^2}{\sigma_x^2} = -2\sum_{m=1}^{t}\sum_{j=1}^{n} \frac{a_m\, \alpha_j\, g(-p_j)}{p_j + k_m}$$

Cross-correlation function and covariance

$$\frac{A_{xy}(\theta)}{\sigma_x^2} = g(-k)\,e^{-k\theta}\,u(\theta) + g(k)\,e^{-k|\theta|}\,u(-\theta) + 2k\,u(-\theta)\sum_{i=1}^{n} \frac{\alpha_i}{p_i^2 - k^2}\, e^{-p_i\theta}$$

with $u(\theta)$ = unit function = $\begin{cases} u(\theta) = 0 & \text{for } \theta < 0 \\ u(\theta) = \frac{1}{2} & \text{for } \theta = 0 \\ u(\theta) = 1 & \text{for } \theta > 0 \end{cases}$

$$\frac{\sigma_{xy}}{\sigma_x^2} = g(-k)$$

3.

Example 1

$$g(p) = -\frac{p_1}{p-p_1}$$

with $\alpha_1 = -p_1$

$$\frac{A_{yy}}{\sigma_x^2} = \frac{p_1^2\, e^{k|\theta|}}{p_1^2 - k^2} - \frac{k\, p_1^2}{p_1(p_1^2-k^2)}\, e^{p_1|\theta|}$$

$$\frac{\sigma_y^2}{\sigma_x^2} = \frac{p_1}{p_1+k}$$

Example 2

$$g(p) = \frac{p_1 p_2}{(p_1-p)(p_2-p)}$$

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = \frac{(p_1p_2)^2 e^{k|\theta|}}{(p_1^2-k^2)(p_2^2-k^2)} + \frac{k\,(p_1p_2)^2 e^{p_1|\theta|}}{p_1(p_1^2-k^2)(p_1^2-p_2^2)} + \frac{k\,(p_1p_2)^2 e^{p_2|\theta|}}{p_2(p_2^2-k^2)(p_2^2-p_1^2)}$$

$$\frac{\sigma_y^2}{\sigma_x^2} = \frac{p_1p_2}{(p_1+k)(p_2+k)}\left(1+\frac{k}{p_1+p_2}\right)$$

Example 3

$$g(p) = -\frac{p_1p_2p_3}{(p-p_1)(p-p_2)(p-p_3)}$$

$$\frac{A_{yy}(\theta)}{\sigma_x^2} = \frac{(p_1p_2p_3)^2 e^{k|\theta|}}{(p_1^2-k^2)(p_2^2-k^2)(p_3^2-k^2)} - k\left\{\frac{(p_1p_2p_3)^2 e^{p_1|\theta|}}{p_1(p_1^2-k^2)(p_1^2-p_2^2)(p_1^2-p_3^2)} + \frac{(p_1p_2p_3)^2 e^{p_2|\theta|}}{p_2(p_2^2-k^2)(p_2^2-p_1^2)(p_2^2-p_3^2)} - \frac{(p_1p_2p_3)^2 e^{p_3|\theta|}}{p_3(p_3^2-k^2)(p_3^2-p_1^2)(p_3^2-p_2^2)}\right\}$$

$$\frac{\sigma_y^2}{\sigma_x^2} = \frac{p_1p_2p_3}{(p_1+k)(p_2+k)(p_3+k)}\left[1 + \frac{k^2(p_1+p_2+p_3) + k(p_1+p_2+p_3)^2}{(p_1+p_2)(p_1+p_3)(p_2+p_3)}\right]$$

These examples open the way to a generalization with a transfer function, with a number n of roots.

3. STOCHASTIC IDENTIFICATION

The use of general relations given above for the computation of the correlation function, variance and covariance of the output, is a useful tool for the identification and the optimization of a system.

In many cases, it is possible, through physical considerations, to determine the structure of the transfer function and some of the elements of this structure ; but very often, the numerical value of some of the parameters are unknown and are to be identified.

If it has been possible to measure $A_{xx}, A_{xy}, A_{xy}, \sigma_x^2, \sigma_y, \sigma_{xy}$ from recorded samples, the identification procedure is the following :

1) Compute the coefficients of the polynomials $N(p)$ and $D(p)$ of the transfer function $g(p) = N(p)/D(p)$ for different values of the parameters to be identified.

2) Compute the corresponding roots p_i and the residues α_i .

3) Use the above given relations compute $A_{yy}, A_{xy}, \sigma_y, \sigma_{xy}$

4) By identification of these functions to the corresponding functions obtained from the measured samples, it is possible to determine the unknown parameters.

This approach makes us of a double functional transformation : from the coefficients of the transfer function to the roots and residues of this function and from these roots and residues to the correlation, variance and covariance. The volume of the computation is very high if the number of the parameters to be identified is great ; but if this number is reduced to one or two, this approach is very efficient.

Example

Let us consider a feed-back system with the block diagram given by fig. 1 and the following transfer function :

$$g(p) = \frac{a(1+pT_3)}{p^3 T_1 T_2 T_3 + p^2 (T_1 T_2 + T_1 T_3 + T_2 T_3) + p(T_1 + T_2 + T_3) + a + 1}$$

We assume that $a = 10$

$T_3 = 0.5$

and that $A_x(\theta) = \sigma_x^2 e^{-k|\theta|}$

We assume that $k = 1$

and that T_1 and T_2 are the parameters which are to be identified.

Using the relations given above σ_y^2/σ_x^2 and σ_{xy}/σ_x^2 have been computed for different values of T_1 and T_2 and the result is given by fig. 2.

Dotted lines correspond to σ_{xy}/σ_x^2 and the plain lines to σ_y^2/σ_x^2.

The area (A) corresponds to an unstable state of the system.

For any measured value of the variance σ_y^2 and the covariance σ_{xy}, it is possible to determine immediately the value of T_1 and T_2, which are to be identified.

CONCLUSION

The given general relations for the computation of the auto and cross-correlation function or the variance and covariance of the output of a linear system give the possibility of identifying the unknown parameters of a system and to analyse the sensitivity of the system, with regard to the variation of these parameters.

On the other hand, by differentiation of the meansquare value of the output, with regard to the parameters of the system which can be adjusted and by putting this differential equal to zero, it is possible to minimize this meansquare value and to optimize the system.

The double functional transformation requires to introduce the product of the different functions, but a general expression of the roots of a polynomial higher than two is unknown. It means that this method can be used only in a given special case and this approach requires further investigation.

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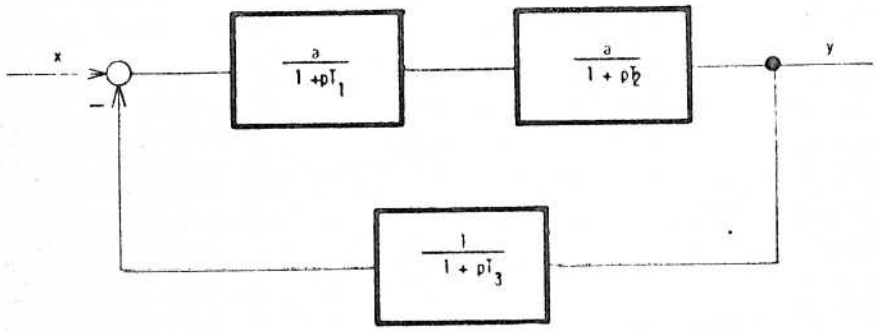
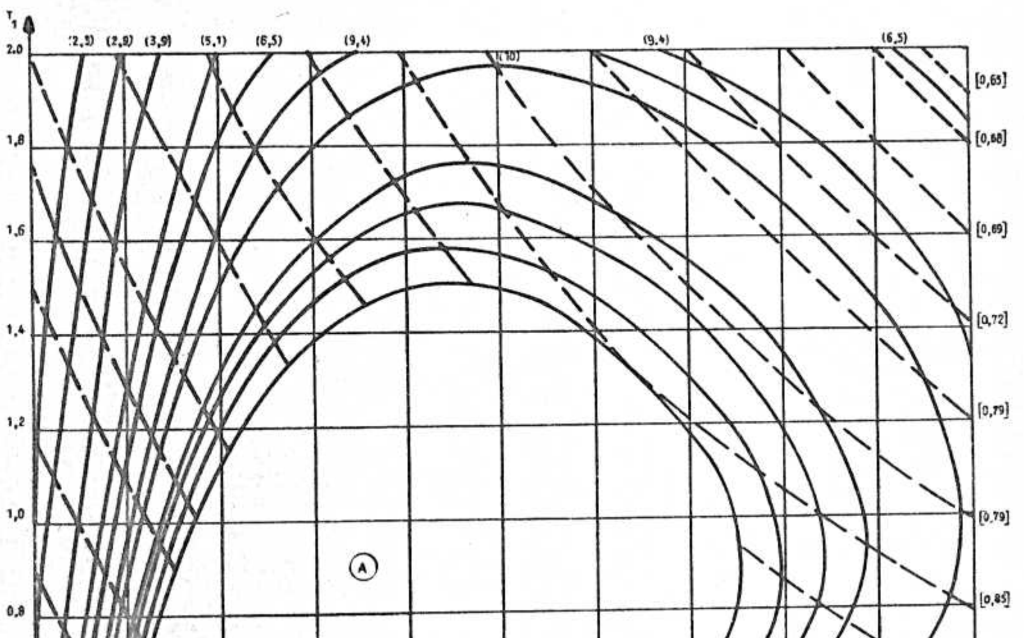


Figure 1 - Example of feedback system .



Heure 2

— $G^2_y\, G^2_x$

--- $G^2_{xy}\, G^2_x$

(A) = unstatile state